# Computing the homology of the real projective plane using cubical theory

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The real projective plane $\mathbf{R}\mathrm{P}^2$ is defined as the space of lines in $\mathbf{R^3}$ passing through the origin. Historically, this space arose from the study of perspective by artists during the Renaissance; today projective geometry is one of the major non-Euclidean geometries. Topologically, $\mathbf{R}\mathrm{P}^2$ is a simple example of a non-orientable manifold (see [4]).

One representation of the topology of $\mathbf{R}\mathrm{P}^2$ is as the quotient space of the closed disk with identification of antipodal points on the boundary; this representation is particular suitable for computing homology. For example, in [2] a CW complex based on this representation is used to calculate

$$H_k(\mathbf{R}\mathrm{P}^2) = \begin{cases} \mathbf{Z}, & \text{for } k = 0, \\ \mathbf{Z}_2, & \text{for } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, this representation admits a triangulation, shown in Figure 1 (see [3]). The following results from [1] show how to use a triangulation in order to construct a homeomorphism between a polyhedron and a cubical set; thus the homology of $\mathbf{R}\mathrm{P}^2$ can be computed using cubical homology:

**Theorem 1** *Let $P$ be a polyhedron, $\mathcal{S}$ a triangulation of $P$, and*

$$\mathcal{V} = \{v_0, v_1, \dots, v_d\}$$

*be the set of vertices in $\mathcal{S}$. For any $n$-simplex $S = \operatorname{conv}\{v_{p_0}, v_{p_1}, \dots, v_{p_n}\} \in \mathcal{S}$, define $f_S\colon S \to \Delta^d$ by*

$$f_S(\textstyle\sum \lambda_i v_{p_i}) := \textstyle\sum \lambda_i \mathbf{e}_{p_i},$$

*where $\lambda_i$ are the barycentric coordinates of a point in $S$, $\mathbf{e}_j$ are the canonical basis vectors of $\mathbf{R^d}$ for $j = 1, 2, \dots, d$, $\mathbf{e}_0 = 0$, and $\Delta^d = \operatorname{conv}\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the standard $d$-simplex. Then extending the maps $f_S$ to a map $f\colon P \to f(P) \subset \Delta^d$ gives a homeomorphism between $P$ and $f(P)$. Moreover, $f$ maps simplices in $\mathcal{S}$ onto simplices of $f(P)$.*

**Theorem 2** *Define $g\colon \Delta^d \to [0,1]^d$ by*

$$g(x) := \begin{cases} 0, & \text{if } x = 0, \\ \frac{x_1 + x_2 + \cdots + x_d}{\max\{x_1, x_2, \dots, x_d\}} & \text{otherwise} \end{cases}$$

*for $x = (x_1, x_2, \dots, x_d)$. Then $g$ is a homeomorphism; moreover, $g$ maps simplices onto cubical subsets of $[0,1]^d$.*

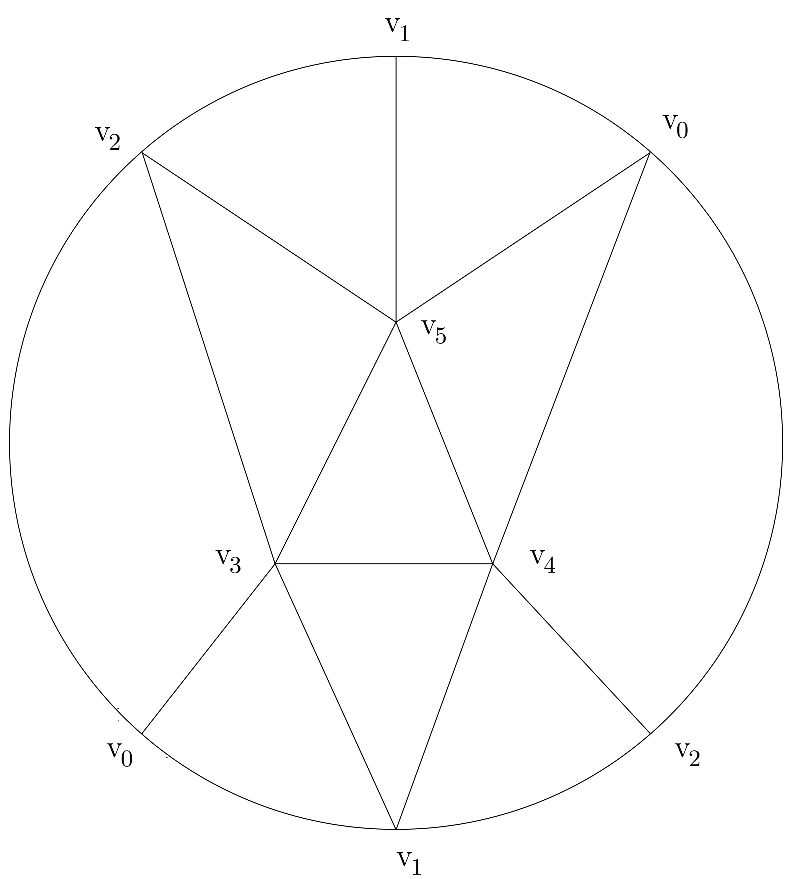


Figure 1: Representing $\mathbf{R}P^2$ as a closed disk with antipodal points of the boundary identified reveals a triangulation.

**Corollary 3** *$h := g \circ f$ is a homeomorphism between $P$ and $h(P)$, where $h(P)$ is a cubical subset of $[0,1]^d$.*

As shown in figure 1, $\mathbf{R}\mathrm{P}^2$ has a triangulation with 6 vertices. Thus it is homeomorphic to a cubical subset $X \in [0,1]^5$. Moreover, the functions in Theorem 1 and Theorem 2 can be used to explicitly compute $X$:

**Proposition 4**

$$\begin{aligned}
h(\mathbf{R}\mathrm{P}^2) \quad = \quad & ([0,1] \times [0] \times [0] \times [0,1] \times [0]) \\
& \cup([0,1] \times [0] \times [0] \times [0] \times [0,1]) \\
& \cup([0] \times [0,1] \times [0,1] \times [0] \times [0]) \\
& \cup([0] \times [0,1] \times [0] \times [0,1] \times [0]) \\
& \cup([0] \times [0] \times [0,1] \times [0] \times [0,1]) \\
& \cup([0,1] \times [0,1] \times [1] \times [0] \times [0]) \\
& \cup([0,1] \times [1] \times [0,1] \times [0] \times [0]) \\
& \cup([1] \times [0,1] \times [0,1] \times [0] \times [0]) \\
& \cup([0,1] \times [0,1] \times [0] \times [0] \times [1]) \\
& \cup([0,1] \times [1] \times [0] \times [0] \times [0,1]) \\
& \cup([1] \times [0,1] \times [0] \times [0] \times [0,1]) \\
& \cup([0,1] \times [0] \times [0,1] \times [1] \times [0]) \\
& \cup([0,1] \times [0] \times [1] \times [0,1] \times [0]) \\
& \cup([1] \times [0] \times [0,1] \times [0,1] \times [0]) \\
& \cup([0] \times [0,1] \times [0] \times [0,1] \times [1]) \\
& \cup([0] \times [0,1] \times [0] \times [1] \times [0,1]) \\
& \cup([0] \times [1] \times [0] \times [0,1] \times [0,1]) \\
& \cup([0] \times [0] \times [0,1] \times [0,1] \times [1]) \\
& \cup([0] \times [0] \times [0,1] \times [1] \times [0,1]) \\
& \cup([0] \times [0] \times [1] \times [0,1] \times [0,1]).
\end{aligned}$$

**Proof** Observe that $h(\mathbf{R}\mathrm{P}^2) = \bigcup\{h(S) : S \text{ is a face of } \mathbf{R}\mathrm{P}^2\}$. Thus it is sufficient to compute the image of each face of $\mathbf{R}\mathrm{P}^2$.

Consider $S = \mathrm{conv}\{v_0, v_1, v_4\}$. For any point $\lambda_0 v_0 + \lambda_1 v_1 + \lambda_4 v_4 \in S$:

$$h(\lambda_0 v_0 + \lambda_1 v_1 + \lambda_4 v_4) \quad = \quad g \circ f_S(\lambda_0 v_0 + \lambda_1 v_1 + \lambda_4 v_4)$$

$$
\begin{aligned}
&= g(\lambda_0\mathbf{e}_0 + \lambda_1\mathbf{e}_1 + \lambda_4\mathbf{e}_4) \\
&= g(\lambda_1\mathbf{e}_1 + \lambda_4\mathbf{e}_4) \\
&= \frac{\lambda_1}{\max(\lambda_1, \lambda_4)}(\lambda_1 + \lambda_4)\mathbf{e}_1 \\
&\quad + \frac{\lambda_4}{\max(\lambda_1, \lambda_4)}(\lambda_1 + \lambda_4)\mathbf{e}_4 \\
&\in [0,1] \times [0] \times [0] \times [0,1] \times [0],
\end{aligned}
$$

because $0 \leq \lambda_1 + \lambda_4 \leq 1$ as $\lambda_0 + \lambda_1 + \lambda_4 = 1$ with $\lambda_i \geq 0$, and $0 \leq \frac{\lambda_i}{\max(\lambda_1, \lambda_4)} \leq 1$.

On the other hand, for $x_1\mathbf{e}_1 + x_4\mathbf{e}_4 \in [0,1] \times [0] \times [0] \times [0,1] \times [0]$: Define $\lambda_1 = \frac{x_1 \max(x_1, x_4)}{x_1 + x_4}$, $\lambda_4 = \frac{x_4 \max(x_1, x_4)}{x_1 + x_4}$, and $\lambda_0 = 1 - \lambda_1 - \lambda_4$. Note that

$$
\begin{aligned}
\lambda_0 &= 1 - \frac{x_1 \max(x_1, x_4)}{x_1 + x_4} - \frac{x_4 \max(x_1, x_4)}{x_1 + x_4} \\
&= 1 - \frac{(x_1 + x_4)\max(x_1, x_4)}{x_1 + x_4} = 1 - \max(x_1, x_4) \geq 0,
\end{aligned}
$$

so $\lambda_0 v_0 + \lambda_1 v_1 + \lambda_4 v_4 \in S$. Then

$$
\begin{aligned}
h(\lambda_0 v_0 + \lambda_1 v_1 + \lambda_4 v_4) &= \frac{\lambda_1}{\max(\lambda_1, \lambda_4)}(\lambda_1 + \lambda_4)\mathbf{e}_1 \\
&\quad + \frac{\lambda_4}{\max(\lambda_1, \lambda_4)}(\lambda_1 + \lambda_4)\mathbf{e}_4 \\
&= \frac{\frac{x_1 \max(x_1, x_4)}{x_1 + x_4}}{\frac{\max(x_1, x_4)^2}{x_1 + x_4}} \max(x_1, x_4)\mathbf{e}_1 \\
&\quad + \frac{\frac{x_4 \max(x_1, x_4)}{x_1 + x_4}}{\frac{\max(x_1, x_4)^2}{x_1 + x_4}} \max(x_1, x_4)\mathbf{e}_4 \\
&= x_1\mathbf{e}_1 + x_4\mathbf{e}_4.
\end{aligned}
$$

Thus $h(S) = [0,1] \times [0] \times [0] \times [0,1] \times [0]$.

By analogous arguments,

$$
\begin{aligned}
h(\mathrm{conv}\{v_0, v_1, v_5\}) &= [0,1] \times [0] \times [0] \times [0] \times [0,1] \\
h(\mathrm{conv}\{v_0, v_2, v_3\}) &= [0] \times [0,1] \times [0,1] \times [0] \times [0] \\
h(\mathrm{conv}\{v_0, v_2, v_4\}) &= [0] \times [0,1] \times [0] \times [0,1] \times [0] \\
h(\mathrm{conv}\{v_0, v_3, v_5\}) &= [0] \times [0] \times [0,1] \times [0] \times [0,1]
\end{aligned}
$$

Now consider $S = \text{conv}\{v_1, v_2, v_3\}$. For any point $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \in S$:

$$\begin{aligned}
h(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) &= g \circ f_S(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) \\
&= g(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3) \\
&= \frac{\lambda_1}{\max(\lambda_1, \lambda_2, \lambda_3)}(\lambda_1 + \lambda_2 + \lambda_3)\mathbf{e}_1 \\
&\quad + \frac{\lambda_2}{\max(\lambda_1, \lambda_2, \lambda_3)}(\lambda_1 + \lambda_2 + \lambda_3)\mathbf{e}_2 \\
&\quad + \frac{\lambda_3}{\max(\lambda_1, \lambda_2, \lambda_3)}(\lambda_1 + \lambda_2 + \lambda_3)\mathbf{e}_3 \\
&= \frac{\lambda_1}{\max(\lambda_1, \lambda_2, \lambda_3)}\mathbf{e}_1 + \frac{\lambda_2}{\max(\lambda_1, \lambda_2, \lambda_3)}\mathbf{e}_2 \\
&\quad + \frac{\lambda_3}{\max(\lambda_1, \lambda_2, \lambda_3)}\mathbf{e}_3
\end{aligned}$$

Thus,

$$h(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) \in \begin{cases} [0,1] \times [0,1] \times [1] \times [0] \times [0], & \text{if } \lambda_3 = \max(\lambda_i), \\ [0,1] \times [1] \times [0,1] \times [0] \times [0], & \text{if } \lambda_2 = \max(\lambda_i), \\ [1] \times [0,1] \times [0,1] \times [0] \times [0], & \text{if } \lambda_1 = \max(\lambda_i). \end{cases}$$

On the other hand, for $\mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \in [1] \times [0,1] \times [0,1] \times [0] \times [0]$: Define $\lambda_1 = \frac{1}{1+x_2+x_3}$, $\lambda_2 = x_2 \lambda_1 \le \lambda_1$, $\lambda_3 = x_3 \lambda_1 \le \lambda_1$. Then:

$$\begin{aligned}
h(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) &= \frac{\lambda_1}{\max(\lambda_1, \lambda_2, \lambda_3)}\mathbf{e}_1 + \frac{\lambda_2}{\max(\lambda_1, \lambda_2, \lambda_3)}\mathbf{e}_2 \\
&\quad + \frac{\lambda_3}{\max(\lambda_1, \lambda_2, \lambda_3)}\mathbf{e}_3 \\
&= \frac{\lambda_1}{\lambda_1}\mathbf{e}_1 + \frac{\lambda_2}{\lambda_1}\mathbf{e}_2 + \frac{\lambda_3}{\lambda_1}\mathbf{e}_3 \\
&= \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3
\end{aligned}$$

Similarly, for $x_1 \mathbf{e}_1 + \mathbf{e}_2 + x_3 \mathbf{e}_3 \in [0,1] \times [1] \times [0,1] \times [0] \times [0]$:

$$h(\frac{x_1}{x_1 + 1 + x_3} v_1 + \frac{1}{x_1 + 1 + x_3} v_2 + \frac{x_3}{x_1 + 1 + x_3} = x_1 \mathbf{e}_1 + \mathbf{e}_2 + x_3 \mathbf{e}_3$$

And for $x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \mathbf{e}_3 \in [0,1] \times [0,1] \times [1] \times [0] \times [0]$:

$$h(\frac{x_1}{x_1 + x_2 + 1} v_1 + \frac{1}{x_1 + x_2 + 1} v_2 + \frac{1}{x_1 + x_2 + 1} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \mathbf{e}_3$$

It follows that

$$\begin{aligned} h(S) &= ([0,1]\times[0,1]\times[1]\times[0]\times[0]) \\ &\cup([0,1]\times[1]\times[0,1]\times[0]\times[0]) \\ &\cup([1]\times[0,1]\times[0,1]\times[0]\times[0]). \end{aligned}$$

By analogous arguments,

$$\begin{aligned} h(\mathrm{conv}\{v_1,v_2,v_5\}) &= ([0,1]\times[0,1]\times[0]\times[0]\times[1]) \\ &\cup([0,1]\times[1]\times[0]\times[0]\times[0,1]) \\ &\cup([1]\times[0,1]\times[0]\times[0]\times[0,1]) \\ h(\mathrm{conv}\{v_1,v_3,v_4\}) &= ([0,1]\times[0]\times[0,1]\times[1]\times[0]) \\ &\cup([0,1]\times[0]\times[1]\times[0,1]\times[0]) \\ &\cup([1]\times[0]\times[0,1]\times[0,1]\times[0]) \\ h(\mathrm{conv}\{v_2,v_4,v_5\}) &= ([0]\times[0,1]\times[0]\times[0,1]\times[1]) \\ &\cup([0]\times[0,1]\times[0]\times[1]\times[0,1]) \\ &\cup([0]\times[1]\times[0]\times[0,1]\times[0,1]) \\ h(\mathrm{conv}\{v_3,v_4,v_5\}) &= ([0]\times[0]\times[0,1]\times[0,1]\times[1]) \\ &\cup([0]\times[0]\times[0,1]\times[1]\times[0,1]) \\ &\cup([0]\times[0]\times[1]\times[0,1]\times[0,1]). \end{aligned}$$

Using the `homcubes` program from the CHomP homology software package, the homology of this cubical representation of $\mathbf{R}\mathrm{P}^2$ was determined to be

$$H_k(\mathbf{R}\mathrm{P}^2) = \begin{cases} \mathbf{Z}, & \text{for } k=0, \\ \mathbf{Z}_2, & \text{for } k=1, \\ 0 & \text{otherwise,} \end{cases}$$

agreeing with the result in [2].

# Appendix: Data for `homcubes`

```
Type of set : C
Space Dimension : 5
dimension 5

;Image of conv{v_0, v_1, v_4}
[0,1] x [0] x [0] x [0,1] x [0]

;Image of conv{v_0, v_1, v_5}
[0,1] x [0] x [0] x [0] x [0,1]

;Image of conv{v_0, v_2, v_3}
[0] x [0,1] x [0,1] x [0] x [0]

;Image of conv{v_0, v_2, v_4}
[0] x [0,1] x [0] x [0,1] x [0]

;Image of conv{v_0, v_3, v_5}
[0] x [0] x [0,1] x [0] x [0,1]

;Image of conv{v_1, v_2, v_3}
[1] x [0,1] x [0,1] x [0] x [0]
[0,1] x [1] x [0,1] x [0] x [0]
[0,1] x [0,1] x [1] x [0] x [0]

;Image of conv{v_1, v_2, v_5}
[1] x [0,1] x [0] x [0] x [0,1]
[0,1] x [1] x [0] x [0] x [0,1]
[0,1] x [0,1] x [0] x [0] x [1]

;Image of conv{v_1, v_3, v_4}
[1] x [0] x [0,1] x [0,1] x [0]
[0,1] x [0] x [1] x [0,1] x [0]
[0,1] x [0] x [0,1] x [1] x [0]

;Image of conv{v_2, v_4, v_5}
[0] x [1] x [0] x [0,1] x [0,1]
```

```
[0] x [0,1] x [0] x [1] x [0,1]
[0] x [0,1] x [0] x [0,1] x [1]

;Image of conv{v_3, v_4, v_5}
[0] x [0] x [1] x [0,1] x [0,1]
[0] x [0] x [0,1] x [1] x [0,1]
[0] x [0] x [0,1] x [0,1] x [1]
```

# Bibliography


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