A GAP package for braid orbit computation, and applications

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Abstract: Let G be a finite group. By Riemann's Existence Theorem, braid orbits of generating systems of G with product 1 correspond to irreducible families of covers of the Riemann sphere with monodromy group G. Thus many problems on algebraic curves require the computation of braid orbits. In this paper we describe an implementation of this computation. We discuss several applications, including the classification of irreducible families of indecomposable rational functions with exceptional monodromy group.

0 Introduction

Let G be a finite group and $\sigma = (\sigma_1, ..., \sigma_r)$ a tuple of elements of G with $\sigma_1 \cdots \sigma_r = 1$. The **braid orbit** of σ is the smallest set of tuples from G that contains σ and is closed under the braid operations

$$(1) \qquad (g_1, ..., g_r)^{Q_i} \quad = \quad (g_1, \ldots,\ g_{i+1},\ g_{i+1}^{-1} g_i g_{i+1}\ , \ldots, g_r)$$

for $i = 1, ..., r-1$. Clearly, the unordered collection of conjugacy classes $C_1, ..., C_r$ represented by the elements of the tuple is an invariant of the braid orbit. This paper describes a package of programs written in [GAP4] for the computation of all braid orbits associated with given classes $C_1, ..., C_r$. We call it the BRAID program. It is available at http://www.math.ufl.edu~helmut. An alternative approach has recently been worked out by Klüners (Kassel), using MAGMA. A precursor was the HO-program of Przywara [Prz] which is now outdated.

Our interest in computing braid orbits comes from the fact that they correspond to irreducible families of covers of the Riemann sphere. This is a classical fact, used by Hurwitz (who found formula (1)) and many algebraic geometers since then. This connection to geometry is briefly explained in section 5. The version required for the application to the Inverse Galois Problem was worked out by Fried and Völklein [FV].

However, there are also purely group-theoretic applications of our braid program, e.g., to find generators of a given group with prescribed element orders. Most applications have been in geometry and number theory, though, via the connection to covers. Covers of $\mathbb{P}^1$ defined over $\mathbb{Q}$ yield Galois realizations of G over $\mathbb{Q}$ via Hilbert's irreducibility theorem — the braid program is needed to find suitable covers for which the criteria of Inverse Galois Theory apply. A good example of that is Malle's construction [Ma] of multi-parameter polynomials with various small Galois groups. His $L_3(2)$-polynomial is used as an example in section 8 below to obtain a generic rational function

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of degree 7 with monodromy group $L_3(2)$. Another example is Matzat's realization [MM],III, 7.5, of M_{24}, for which Granboulan [Gra] computed an explicit polynomial. A further example is the realization of symplectic groups $Sp(n, q)$ by Thompson and Völklein [ThV] which depends on the fact that the pure braid operations (2) generate an abelian group of permutations of the corresponding braid orbit (mod conjugation). There are numerous other applications to the Inverse Galois Problem, see [MM] and [V1].

There are also applications to problems about the geometry of algebraic curves and their moduli spaces $\mathcal{M}_g$. E.g., in [MSSV] the authors study the locus in $\mathcal{M}_g$ of curves with given 'large' automorphism group G. The irreducible components of that locus correspond to certain braid orbits in G. The BRAID program enabled us to completely classify these components for $g \leq 10$ and compute the genus of those that are 1-dimensional.

In this paper we describe the application to classifying the irreducible families of indecomposable rational functions with monodromy group other than S_n or A_n. A generating system $\sigma_1, \ldots, \sigma_r$ of a transitive permutation group G with $\sigma_1 \cdots \sigma_r = 1$ is called a **genus zero system** if the corresponding covers of $\mathbb{P}^1$ have genus 0, i.e., are given by a rational function $f(x) \in \mathbb{C}(x)$. The function f is **indecomposable** (with respect to composition) if and only if G is primitive. In this case we say $\sigma_1, \ldots, \sigma_r$ is a **primitive genus zero system**. There is a huge variety of such systems that generate S_n or A_n, too many to be classified. Those functions with smaller monodromy group satisfy interesting identities and therefore it seems desirable to have a complete classification of their irreducible families.

Thus we need to compute all braid orbits of genus zero systems in primitive permutation groups G other than A_n or S_n. It follows from the proof of the Guralnick-Thompson Conjecture (see [FM]) that only finitely many groups G occur. The complete list is being worked out by Frohardt, Guralnick, Magaard and Shareshian [FGM2], [GS] (project nearly completed). The smallest group that occurs is $G = L_3(2)$ (acting on 7 points). We study this example in section 8. In section 9 we present all braid orbits of genus zero systems of length ≥ 5 in almost simple groups other than A_n or S_n. The remaining cases (length 3 and 4) will be collected in a data base, there is too many of them to be displayed here.

Another application of the BRAID program was given in [MV]. We say a tuple $\sigma_1, \ldots, \sigma_r$ in S_n has **full moduli dimension** if the corresponding family of covers contains the general curve of that genus. If that holds and the genus is at least 4 then $\sigma_1, \ldots, \sigma_r$ generate S_n or A_n by work of Guralnick and others [GM], [GS]. In genus 2 and 3 there are several other possible cases. In [MV] it was shown that the general curve of genus 3 has a cover to $\mathbb{P}^1$ of degree 7 with monodromy group $L_3(2)$. The associated tuple consists of 9 involutions (with product 1) generating $L_3(2)$. There is only one braid orbit of such tuples by [MV], Remark 5.1. This requires an iterative application of the BRAID program because the orbit is too large for a direct computation. This iterative procedure for computing braid-orbits of long tuples in small groups requires computing braid-orbits of (shorter) tuples of product $\neq 1$ (see Remark 1.1).

Part I
Description of the BRAID program

1 Exact formulation of the problem

Fix an integer $r \geq 3$.

The Artin braid group $\mathcal{B}_r$ is defined by a presentation on generators $Q_1, ..., Q_{r-1}$ and relations

$$Q_i Q_{i+1} Q_i \;=\; Q_{i+1} Q_i Q_{i+1} \quad \text{and} \quad Q_i Q_j \;=\; Q_j Q_i \;\; \text{for } |i-j| > 1$$

Mapping Q_i to the transposition $(i, i+1)$ extends to a homomorphism $\kappa : \mathcal{B}_r \to S_r$ with kernel $\mathcal{B}^{(r)}$, the pure Artin braid group. It is generated by the

$$(2) \quad Q_{ij} \;=\; Q_{j-1} \cdots Q_{i+1}\, Q_i^2\, Q_{i+1}^{-1} \cdots Q_{j-1}^{-1} \;=\; Q_i^{-1} \cdots Q_{j-2}^{-1}\, Q_{j-1}^2\, Q_{j-2} \cdots Q_i, \quad 0 \leq i < j \leq r$$

More generally, if P is a partition of $\{1, \ldots, r\}$, let S_P be the stabilizer of P in S_r and set $\mathcal{B}_P = \kappa^{-1}(S_P)$. We always choose P such that each block consists of all integers between the smallest and largest element of the block. Thus we can identify P with the list of the lengths of its parts. $\mathcal{B}_P$ is generated by the Q_{ij} with i, j not in the same block of P, and the Q_i with $i, i+1$ in the same block.

Now let G be a finite group. Then $\mathcal{B}_r$ acts on r-tuples of elements of G with product 1 via formula (1) above. The orbits of this $\mathcal{B}_r$-action are called **braid orbits**. This $\mathcal{B}_r$-action commutes with the action of $\mathrm{Aut}(G)$ on tuples defined by

$$\alpha(\sigma_1, \ldots, \sigma_r) \;=\; (\alpha(\sigma_1), \ldots, \alpha(\sigma_r))$$

for $\alpha \in \mathrm{Aut}(G)$. Thus $\mathcal{B}_r$ permutes $\mathrm{Aut}(G)$-orbits (as well as $\mathrm{Inn}(G)$-orbits) of tuples.

Note that in the $\mathcal{B}_r$-action on tuples $(\sigma_1, \ldots, \sigma_r)$, the conjugacy classes $\sigma_1^G, ..., \sigma_r^G$ are being permuted via the map $\kappa : \mathcal{B}_r \to S_r$. This yields an obvious simplification in computing the braid orbit of a tuple $(\sigma_1, \ldots, \sigma_r)$: We only need to compute those tuples in the braid orbit where the classes $\sigma_1^G, ..., \sigma_r^G$ occur in that given order. In other words, we only compute the orbit of $(\sigma_1, \ldots, \sigma_r)$ under the subgroup of $\mathcal{B}_r$ that stabilizes this order of the conjugacy classes. This subgroup equals $\mathcal{B}_P$, where P is the partition of $\{1, \ldots, r\}$ such that i and j lie in the same block iff σ_i is conjugate σ_j.

The classes $\sigma_1^G, ..., \sigma_r^G$ have an important interpretation in terms of the associated covers ("distinguished inertia group generators", see [V1]). Thus we consider the following basic problem.

Problem 1: *Let $C_1, ..., C_r$ be non-trivial conjugacy classes of the finite group G. Let P be the partition of $\{1, \ldots, r\}$ such that i and j lie in the same block iff $C_i = C_j$. We want to compute the orbits of $\mathcal{B}_P$ on the set of $\mathrm{Inn}(G)$-orbits on*

$$\mathfrak{E}(C_1, ..., C_r) \;=\; \{(\sigma_1, \ldots, \sigma_r) : \; \sigma_i \in C_i, \; \sigma_1 \cdots \sigma_r = 1\}$$

Further geometric information is furnished by the permutations induced by certain of the generators of $\mathcal{B}_P$ on the braid orbit. So we record these permutations as we construct the braid orbit. In the case $r = 4$, for example, this information can be used to compute the genus of the corresponding Hurwitz curve (see section 6 below).

Remark 1.1 *Modified versions of Problem 1 arise where $\mathcal{B}_P$ is replaced by a subgroup $\mathcal{B}'$. For example, $\mathcal{B}'$ could be $\mathcal{B}_{P'}$ for a partition P' finer than P, or it could be an analogous subgroup of $\mathcal{B}_{r-1}$. The latter is equivalent to acting on tuples of length $r-1$ with product $\neq 1$. (Note that the braid group acts on tuples with any fixed product by formula (1)). Further choices for $\mathcal{B}'$ are the subgroups of the braid group induced by the fundamental groups of certain curves on the configuration space, see [De]; generators for some of these groups can be found at http://www.iwr.uni-heidelberg.de/groups/compalg/dettweil/papers.html. (They have applications to the Inverse Galois Problem). The BRAID program can easily be adapted to these modified versions of Problem 1.*

2 Program input and output

Problem 1 is solved by our main routine `AllBraidOrbits`. To call this routine, choose a tuple τ representing the classes $C_1, ..., C_r$. (The tuple τ need not have product 1). The classes $C_1, ..., C_r$ must be ordered such that if $C_i = C_j$ with $i < j$ then $C_i = C_k$ for all $i \leq k \leq j$. The cardinality c of $\mathfrak{E}(C_1, ..., C_r)$ is given by a well-known formula (see [MM], Ch.I, Th. 5.8) involving the values on $C_1, ..., C_r$ of the irreducible characters of G. This number c is called the **structure constant** associated with $C_1, ..., C_r$. It can be computed with the GAP command `ClassStructureCharTable`, once the character table of G is available. Once c has been computed, we call our main routine in the form

$$\texttt{AllBraidOrbits("ProjectName",}G,\tau,P,c\texttt{)}$$

where `ProjectName` is any string that is used to label the output files. Here G has to be a permutation group because many standard algorithms of GAP4 work only in that case. The routine computes the $\mathcal{B}_P$-orbits on $\mathfrak{E}(C_1, ..., C_r)$ mod $\mathrm{Inn}(G)$. For each orbit it creates a file containing a list of representatives of $\mathrm{Inn}(G)$-orbits of the tuples in the orbit, plus the permutations induced on the orbit by the generators of $\mathcal{B}_P$ and by the generators of the pure braid group.

User-friendly version: G and τ are as above. The routine

$$\texttt{Braid(}G\texttt{,}\tau\texttt{)}$$

firstly computes the character table of G and uses it to compute the structure constant c. For large G this may be time-consuming or not feasible at all (then the character table must be taken from some library). Furthermore, the program computes the partition P. Then it calls `AllBraidOrbits`, using always the same ProjectName "TEMP". The previous contents of that directory is removed each time the routine is called. In the end, it summarizes the output by listing all braid orbits found that consist of tuples σ generating G. If $r = 4$, the genus of the inner Hurwitz curve $\mathcal{H}_{\mathrm{in}}^{\mathrm{red}}(\sigma)$ and straight inner Hurwitz curve $\tilde{\mathcal{H}}_{\mathrm{in}}^{\mathrm{red}}(\sigma)$ are given for each of those orbits (see section 5, 6). A variation is the command

$$\texttt{Braid(}G\texttt{,}\tau\texttt{,}U\texttt{)}$$

where U is a core-free subgroup of G of index n. Now the routine calls `AllBraidOrbits` with G replaced by its normalizer in S_n, where G is embedded in S_n via its permutation representation on the cosets of U. If $r = 4$, the genus of the Hurwitz curve $\mathcal{H}^{\mathrm{red}}(\sigma)$ (relative to this permutation representation) is given for each orbit of tuples generating G.

3 Description of the algorithm

At the beginning of its main loop, the `AllBraidOrbits` routine collects a batch of random tuples from $\mathfrak{E}(C_1, \ldots, C_r)$. If one of these tuples does not belong to a known (braid) orbit, a routine `BraidOrbit` is called to generate the new orbit and add it to the list of known orbits. Furthermore, the variable c is adjusted to be the number of tuples in $\mathfrak{E}(C_1, \ldots, C_r)$ which do not belong to any one of the currently known orbits. When $c = 0$, we are done.

One is mainly interested in those tuples from $\mathfrak{E}(C_1, \ldots, C_r)$ that generate G. However, we don't know how to determine their number beforehand (in any efficient way). That's why we are working with the larger set $\mathfrak{E}(C_1, \ldots, C_r)$ (whose cardinality c is given by the structure constant formula). Here are some variations on choosing the input value of c: Setting c to a very large number, `AllBraidOrbits` is turned into an infinite loop. The user breaks the loop when he is convinced that all relevant orbits have been found. This avoids the actual computation of the structure constant. On the other hand, by setting c below the actual size of $\mathfrak{E}(C_1, \ldots, C_r; 1)$ one can skip the last few small orbits that are usually irrelevant. For example, if only the orbits of generating tuples are of interest then one can quit once the number of tuples unaccounted for is below $|G/Z(G)|$.

Hitting a particular small orbit with a random tuple is not likely to happen quickly. Therefore, we implemented a particular way of creating random tuples. It involves maintaining a list of small subgroups generated by known tuples, and trying to find more tuples in those subgroups. For example, the case of 6-tuples of double transpositions in A_7 took about 2 hours using a purely random tuple selection. Our current method cut this time to 30 minutes. In both cases the program took 20 minutes to account for about 90% of the tuples. So the time for finding the last 10% was cut from 100 minutes to 10 minutes.

The routine `BraidOrbit`(σ) constructs the braid orbit of a tuple σ. We use a Dixon-Schreier algorithm: Beginning with σ, apply the generators of $\mathcal{B}_P$ one by one to the known tuples and check wether or not the image is G-conjugate to one them. If not we append the new tuple to the list. The routine terminates when no further tuples can be produced.

The only difficulty is how to check efficiently whether two given tuples are G-conjugate. To speed this up we use a fingerprinting technique. Fingerprints are sequences of numbers that can be quickly computed for a tuple. Tuples with distinct fingerprints cannot be conjugate. Currently, fingerprints are realized as the orders (as group elements) of certain random words in $\sigma_1, \ldots, \sigma_r$. The fingerprints are stored along with the tuples. Access to a tuple is via its fingerprint. Access to a fingerprint is via a hash table, the address for which is formed from the entries of the fingerprint. We remark that this method works well for a large variety of groups G. Exceptions are Frobenius groups and some p-groups.

4 A sample session: Tuples of 4 involutions in S_3

gap> g:=SymmetricGroup(3);;
gap> t:=[$(1,2),(1,2),(1,2),(1,2)$];

gap> Braid(g,t);

Collecting 20 random tuples... done

Cleaning done; 20 random tuples remaining

Orbit 1:
Length=4
Generated subgroup size=6
Centralizer size=1
Remaining portion of structure constant=3
Cleaning current orbit... done; 1 random tuples remaining

Orbit 2:
Length=1
Generated subgroup size=2
Centralizer size=2
Remaining portion of structure constant=0
Cleaning current orbit... done; 0 random tuples remaining

Summary: orbits of generating tuples

Orbit of Length 4
Inner Hurwitz curve genus = 0
Straight inner Hurwitz curve genus = 0

Part II

Applications of the BRAID program

5 Brief explanation of the background on covers

Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. A **cover** of $\mathbb{P}^1$ (in the classical sense) is a compact Riemann surface X together with a non-constant analytic map $f : X \to \mathbb{P}^1$ of finite degree. By Riemann's Existence Theorem, f can also be viewed as a morphism of complex algebraic curves.

Consider such a cover $f : X \to \mathbb{P}^1$ of degree n. It has finitely many branch points $p_1, ..., p_r \in \mathbb{P}^1$ (points whose preimage has cardinality less than n). Pick $p \in \mathbb{P}^1 \setminus \{p_1, ..., p_r\}$, and choose loops γ_i around p_i such that $\gamma_1, ..., \gamma_r$ is a standard generating system of the fundamental group $\Gamma := \pi_1(\mathbb{P}^1 \setminus \{p_1, ..., p_r\}, p)$ (see [V1], Thm. 4.27); in particular, we have $\gamma_1 \cdots \gamma_r = 1$. Such a system $\gamma_1, ..., \gamma_r$ is called a homotopy basis of $\mathbb{P}^1 \setminus \{p_1, ..., p_r\}$. The group Γ acts on the fiber $f^{-1}(p)$ by path lifting, inducing a transitive subgroup G of the symmetric group S_n (determined by f up to conjugacy in S_n). It is called the **monodromy group** of f. The images of $\gamma_1, ..., \gamma_r$ in S_n form a tuple $\sigma = (\sigma_1, ..., \sigma_r)$ generating G. We say the cover $f : X \to \mathbb{P}^1$ is **of type** σ. The genus g of X

depends only on σ, and is given by the **Riemann-Hurwitz formula**

$$2\ (n+g-1) \quad = \quad \sum_{i=1}^{r} \operatorname{Ind}(\sigma_{\mathrm{i}}) \tag{3}$$

where the index $\operatorname{Ind}(\sigma_{\mathrm{i}})$ of a permutation in S_n is n minus the number of orbits.

A tuple $\sigma = (\sigma_1, ..., \sigma_r)$ of elements of S_n arises in the above way from a cover of degree n if and only if σ generates a transitive subgroup G and $\sigma_1 \cdots \sigma_r = 1$ and $\sigma_i \neq 1$ for all i. Call such a tuple **admissible**. The significance of braid orbits comes from the following fact (which follows from Nielsen's theorem).

Theorem: *Let σ and σ' be admissible tuples generating the same subgroup G of S_n. Suppose $f: X \to \mathbb{P}^1$ is a cover of type σ. Then f is of type σ' if and only if the braid orbits of σ and σ' are conjugate under $N_{S_n}(G)/G$.*

Here $N_{S_n}(G)$ is the normalizer of G in S_n. The action of $N_{S_n}(G)/G$ on braid orbits comes from the fact that if σ generates G then $\operatorname{Inn}(G)$ fixes the braid orbit of σ (see [V1], Lemma 9.4).

The next important fact is that the covers of type σ form an *irreducible family.* Here we use the term "family" in the non-technical sense: Two covers are in the same irreducible family if they can be continously deformed into each other (keeping the branch points distinct). It turns out that the covers of type σ are parametrized (up to equivalence) by an irreducible variety, the **Hurwitz space** $\mathcal{H}(\sigma)$. This is made precise in the theory of Hurwitz spaces (= moduli spaces for covers of $\mathbb{P}^1$), see [FV], [V1],[V2].

Two covers $f: X \to \mathbb{P}^1$ and $f': X' \to \mathbb{P}^1$ are called equivalent (resp., weakly equivalent) if there is a homeomorphism $h: X \to X'$ (resp., a homeomorphism $h: X \to X'$ and an analytic automorphism g of $\mathbb{P}^1$) such that $f = f' \circ h$ (resp., $g \circ f = f' \circ h$). The automorphism group of $\mathbb{P}^1$ is $\mathrm{PGL}_2(\mathbb{C})$ (group of fractional linear transformations). It has a natural action on the Hurwitz space $\mathcal{H}(\sigma)$. The quotient by this action is the reduced Hurwitz space $\mathcal{H}^{\mathrm{red}}(\sigma)$. It parametrizes the covers of type σ up to weak equivalence. Summarizing:

Basic Fact: *The covers of type σ are parametrized up to equivalence (resp., up to weak equivalence) by an irreducible variety, the* **Hurwitz space** *$\mathcal{H}(\sigma)$ (resp., $\mathcal{H}^{red}(\sigma)$). These varieties depend only on the braid orbit of σ.*

A cover $f: X \to \mathbb{P}^1$ of type σ is a Galois cover if and only if σ generates a regular subgroup G of S_n. Pairs (f, μ), where f is a Galois cover of type σ and $\mu: \operatorname{Deck}(f) \to G$ an isomorphism, are parametrized by the inner Hurwitz space $\mathcal{H}_{\mathrm{in}}(\sigma)$ (up to suitable equivalence). This also is an irreducible variety. Its quotient by $\mathrm{PGL}_2(\mathbb{C})$ is the inner reduced Hurwitz space $\mathcal{H}_{\mathrm{in}}^{\mathrm{red}}(\sigma)$. It is the inner Hurwitz space that is of foremost importance for the Inverse Galois Problem (see [FV]). There is another version of it, the straight inner Hurwitz space $\tilde{\mathcal{H}}_{\mathrm{in}}(\sigma)$ that parametrizes pairs (f, μ) together with an ordering of the branch points of f. It also has a reduced version $\tilde{\mathcal{H}}_{\mathrm{in}}^{\mathrm{red}}(\sigma)$.

If σ has length $r \leq 3$ then $\mathcal{H}^{\mathrm{red}}(\sigma)$ and $\mathcal{H}_{\mathrm{in}}^{\mathrm{red}}(\sigma)$ consist just of a single point. If $r = 4$ then these reduced Hurwitz spaces are curves. In the next section we show how to compute their genus.

6 The genus of the reduced Hurwitz curve in the case $r = 4$

In this section we look at the case $r = 4$. The braid group $\mathcal{B}_4 =< Q_1, Q_2, Q_3 >$ acts on Inn(G)-orbits of admissible 4-tuples from G via its quotient $\overline{\mathcal{B}}_4$ defined by the extra relations

$$Q_1Q_2Q_3^2Q_2Q_1 \quad = \quad 1 \quad = \quad Q_1^2Q_3^{-2}$$

The structure of $\overline{\mathcal{B}}_4$ has been determined by Thompson [Th]. We denote the image of Q_i in $\overline{\mathcal{B}}_4$ by the same symbol, for simplicity. The elements $\gamma_0 = Q_1Q_2$ and $\gamma_1 = Q_1Q_2Q_1$ of $\overline{\mathcal{B}}_4$ have order 3 and 2, respectively. The elements $Q_1Q_3^{-1}$ and $(Q_1Q_2Q_3)^2$ generate a normal Klein 4-group $\mathcal{V}$ in $\overline{\mathcal{B}}_4$, and $\overline{\mathcal{B}}_4/\mathcal{V}$ is the free product of the images of $< \gamma_0 >$ and $< \gamma_1 >$.

Fix an admissible 4-tuple $\sigma = (\sigma_1, ..., \sigma_4)$, and let $G \subset S_n$ be the group generated by σ. Two 4-sets (unordered 4-tuples) of points of $\mathbb{P}^1$ are $\mathrm{PGL}_2(\mathbb{C})$-conjugate if and only if they have the same j-invariant (which can be any complex number). The covers f of type σ whose branch points have fixed j-invariant $\neq 0, 1$ are parametrized, up to weak equivalence, by the set F of $\mathcal{V}$-orbits of $N_{S_n}(G)$-orbits of 4-tuples in the braid orbit of σ. (Follows from the theory outlined in section 5, plus the fact that the stabilizer in $\mathrm{PGL}_2(\mathbb{C})$ of any 4-set with j-invariant $\neq 0, 1$ is a Klein 4-group). From this one obtains an explicit description of the **Hurwitz curve** $\mathcal{H}^{\mathrm{red}}(\sigma)$ parametrizing the covers of type σ (up to weak equivalence). It arises as covering of $\mathbb{P}^1$ with branch points at $0, 1, \infty$ whose general fiber is in 1-1 correspondence with F. The triple of permutations associated with this covering (by section 5) is given by the action on F of γ_0, γ_1 and $\gamma_\infty := Q_2$ (see [BF], Prop. 4.4 and [DF], Prop. 6.5). From this we can compute the genus of $\mathcal{H}^{\mathrm{red}}(\sigma)$ by the Riemann-Hurwitz formula (2). The case of the inner reduced Hurwitz curve $\mathcal{H}^{\mathrm{red}}_{\mathrm{in}}(\sigma)$ is analogous, with F replaced by the set of $\mathcal{V}$-orbits of Inn(G)-orbits of 4-tuples in the braid orbit of σ.

7 Indecomposable rational functions and primitive genus zero systems

Here we are concerned with covers $f : X \to \mathbb{P}^1$ where X has genus 0. Then we can identify X with $\mathbb{P}^1$, so we consider covers $f : \mathbb{P}^1 \to \mathbb{P}^1$. If such a cover has degree n then it is given by a rational function of degree n, i.e., $f(x) = P(x)/Q(x)$ where P and Q are complex polynomials with $n = \max(\deg(P), \deg(Q))$. Then the monodromy group G of f is isomorphic (as a permutation group) to the Galois group of the polynomial $P(x) - tQ(x)$ over $\mathbb{C}(t)$. By the Riemann-Hurwitz formula (2), genus 0 covers correspond to the following kind of tuples:

Definition 7.1 *A* **genus zero system** *in S_n is a tuple $(\sigma_1, ..., \sigma_r)$ generating a transitive subgroup G of S_n such that $\sigma_1 \cdots \sigma_r = 1$ and $\sigma_i \neq 1$ (for all i) and*

$$2\ (n-1) \quad = \quad \sum_{i=1}^{r} \mathrm{Ind}(\sigma_{\mathrm{i}})$$

It is called a **primitive genus zero system** *if G is primitive.*

Thus by section 5, irreducible families of rational functions in $\mathbb{C}(x)$ of degree n with monodromy group $G \subset S_n$ correspond to $N_{S_n}(G)/G$-orbits of braid orbits of genus zero systems generating

G. The family consists of indecomposable functions if and only if G is primitive. Here "indecomposable" means that the function is not the composition $f_1(f_2(x))$ of two functions of degree > 1.

There is a huge number of genus zero systems that generate S_n or A_n, too many to be classified. The 'general' rational function has monodromy group S_n. Those functions with smaller monodromy group satisfy interesting identities and therefore it seems desirable to have a complete classification of their irreducible families. They correspond to the primitive genus zero systems that generate a permutation group G other than S_n or A_n. The smallest case is $G = L_3(2)$ (acting on 7 points). It has the most braid orbits of genus zero systems. We discuss this example in the following section.

8 Example: Genus zero systems for the action of $G = L_3(2)$ on 7 points

The braid orbits of such tuples are listed in Table 1. We note there is exactly one braid orbit B_6 of tuples of length 6, all the others consist of shorter tuples.

Replacing the last two entries of a tuple σ by their product is called "Coalescing the tuple". Geometrically, this means that we merge (or "coalesce") the last two branch points of the associated cover. The family corresponding to the coalesced tuple σ' lies in the boundary of the original family; in other words, the generic cover of type σ' arises by specialization of the generic cover of type σ.

One checks that each of the orbits $\neq B_6$ in Table 1 contains a tuple that arises by a sequence of such coalescing operations from a tuple of length 6. This means that there is essentially only one family of rational functions of degree 7 with monodromy group $G = L_3(2)$. The generic function in this family has 6 branch points, and on the boundary we have functions with 3, 4 or 5 branch points. We can extract an explicit form of such a generic function from [Ma], Thm. 4.3:

Generic function of degree 7 with monodromy group $L_3(2)$:

$$f(x) \quad = \quad \frac{P(x)}{x^2(x-c)(x^2-bx+b)}$$

where

$$P(x) \;=\; x^7-(a\,(c-2)+2\,b+c)\,x^6+(-(b-4)\,(c-1)\,a^2+((c-2)\,b^2+(2\,c^2-5\,c+4)\,b-2\,c^2)\,a+b\,(2\,b\,c+2\,c^2+b^2))x^4+$$
$$((2\,c^2-1)\,(b-4)\,a^2+((-2\,c^2+c+2)\,b^2+(5\,c^2+2\,c-4)\,b-4\,c^2)\,a-(c+1)\,b^3-c\,(2\,c+3)\,b^2+c^2\,b)x^3$$
$$+((c^2+3\,c-1)\,(4-b)\,a^2+((3\,c-2)\,b^2-2\,(c^2+4\,c-2)\,b+4\,c^2)\,a+b\,(b^2+3\,b\,c-c^2))c\,x^2$$
$$+\,(2\,a\,b\,c-8\,a\,c+a\,b-4\,a-b^2+2\,b\,c)\,a\,c^2\,x-a^2\,(b-4)\,c^3$$

Replacing a function $g(x)$ by $\alpha(g(\beta(x))$ with $\alpha, \beta \in \mathrm{PGL}_2(\mathbb{C})$ doesn't change the monodromy group. So the functions we are interested in are only determined up to coordinate change. (Weak equivalence of covers, see above).

To illustrate the interplay between these functions and the group-theoretic data in Table 1, we consider the specialization $b = 0$. The resulting function $y = h(x)$ still has degree 7. It has poles of order 4,2,1 at $x = 0, \infty, c$, respectively. Thus the corresponding tuple σ contains an element of cycle type (4)(2) (corresponding to the branch point $y = \infty$). Thus the monodromy group of $h(x)$ is still $L_3(2)$ (since it is a transitive subgroup of $L_3(2)$ containing an element of order 4). The ramification index at a point $x = x_0$ not over $y = \infty$ equals one plus the multiplicity of the zero $x = x_0$ of the derivative $h'(x)$. Here we can replace $h'(x)$ by its numerator (when it is written as a rational function in reduced form). This numerator is a lengthy expression of degree 8 in x. But its discriminant with respect to x factors nicely as $16777216\, c^{16}\, a^9$ times the cube of the following expression (4) times the square of another (slightly longer) expression (5) that we don't display here.

$$4\,a^2\,c^4 + 8\,a\,c^4 + 4\,c^4 - 4\,a^2\,c^3 - 36\,c^3\,a + a^3\,c^2 + 6\,a^2\,c^2 + 16\,c^2\,a - 2\,a^3\,c - 8\,c\,a^2 + 2\,a^3 + 16\,a^2 \quad (4)$$

The discriminant is non-zero, hence the above ramification indices are all ≤ 2. It follows that σ consists of an element of order 4 and four involutions (by Riemann-Hurwitz). Thus $h(x)$ is the generic function in the $(2A, 2A, 2A, 2A, 4A)$-family from Table 1.

Let's see how we can further specialize $h(x)$ by coalescing two of the finite branch points. By Table 1, this leads to the $(2A, 2A, 3A, 4A)$- and the $(2A, 2A, 4A, 4A)$-family. Both of those have ramification indices > 1 at certain points not over $y = \infty$. Hence these specializations annihilate the above discriminant. The two factors (4) and (5) define genus zero curves in the a, c-plane (checked by [Maple]). This corresponds nicely to the fact that the $(2A, 2A, 3A, 4A)$- and the $(2A, 2A, 4A, 4A)$-family are parametrized by Hurwitz curves of genus zero (see Table 1).

Incidentally, [Ma], Thm. 4.2 gives another version of the generic function in the $(2A, 2A, 2A, 2A, 4A)$-family. (He doesn't consider our version). One can similarly specialize it to obtain two genus zero curves parametrizing the $(2A, 2A, 3A, 4A)$- and the $(2A, 2A, 4A, 4A)$-family.

Table 1: Genus zero systems for the action of $G = L_3(2)$ on 7 points

classes $C_1, ..., C_r$	length of orbits	number of orbits	genus	straight genus
$(2A, 2A, 2A, 2A, 2A, 2A)$	1680	1		
$(2A, 2A, 2A, 2A, 3A)$	216	1		
$(2A, 2A, 2A, 2A, 4A)$	192	1		
$(2A, 2A, 2A, 7A)$	7	1	0	0
$(2A, 2A, 2A, 7B)$	7	1	0	0
$(2A, 2A, 3A, 3A)$	30	1	0	2
$(2A, 2A, 3A, 4A)$	24	1	0	1
$(2A, 2A, 4A, 4A)$	24	1	0	1
$(2A, 3A, 7A)$	1	1		
$(2A, 3A, 7B)$	1	1		
$(2A, 4A, 7A)$	1	1		
$(2A, 4A, 7B)$	1	1		
$(3A, 3A, 4A)$	1	4		
$(3A, 4A, 4A)$	1	2		
$(4A, 4A, 4A)$	1	4		

9 Primitive genus zero covers branched at ≥ 5 points

Each primitive permutation group has a characteristic subgroup $F^*(G)$ (called the generalized Fitting subgroup) which is the direct product of isomorphic simple groups. Frohardt, Guralnick and Magaard [FGM2] determine all primitive genus zero systems generating a group $G \subset S_n$ with $F^*(G)$ not abelian and not a direct product of alternating groups. The resulting list is finite, but too long to be shown in tabular form. However, there are only a few cases with $r \geq 5$ (i.e., where the corresponding covers are branched at 5 or more points). We list these in Table 2 and note that for each choice of $C_1, ..., C_r$ there is exactly one associated braid orbit (i.e., exactly one irreducible family of genus zero covers).

The table was produced as follows. A series of reductions shows that the permutation degree of such a system is at most 1000. It remains to search the GAP library of primitive permutation groups of degree ≤ 1000. For each such group G that satisfies our hypothesis, we find all collections of conjugacy classes $C_1, ..., C_r$ that satisfy the Riemann-Hurwitz formula (for $g = 0$). For each such collection, we apply the BRAID program to find all braid orbits of associated tuples.

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Table 2: Braid orbits of primitive genus zero systems of length ≥ 5 in almost simple groups $\neq A_n, S_n$

G	degree	classes $C_1, ..., C_r$	orbit length
$L_4(3)$	40	$(2A, 2B, 2B, 2C, 2C)$	320
$S_6(2)$	36	$(2A, 2B, 2B, 2B.3B)$	4
$L_5(2)$	31	$(2B, 2B, 2B, 2B, 2B)$	31744
	31	$(2A, 2A, 2B, 2B, 3B)$	528
$S_6(2)$	28	$(2A, 2A, 2A, 3B, 4A)$	4
	28	$(2A, 2C, 2C, 2C, 3B)$	54
	28	$(2A, 2D, 2D, 2D, 2D)$	3584
M_{24}	24	$(2A, 2A, 2A, 2A, 4B)$	72000
M_{23}	23	$(2A, 2A, 2A, 2A, 3A)$	21456
M_{22}	22	$(2A, 2A, 2A, 2B, 2C)$	660
	22	$(2A, 2A, 2B, 2B, 3A)$	600
$L_3(4)$	21	$(2A, 2A, 2A, 2A, 2A)$	252
$L_3(4).3.2_2$	21	$(2B, 2B, 2B, 2B, 3A)$	1824
	21	$(2A, 2A, 2B, 2B, 3B)$	264
$L_3(3)$	13	$(2A, 2A, 2A, 2A, 2A, 2A)$	32760
	13	$(2A, 2A, 2A, 2A, 3B)$	1944
	13	$(2A, 2A, 2A, 2A, 4A)$	2016
	13	$(2A, 2A, 2A, 2A, 6A)$	2160
	13	$(2A, 2A, 2A, 3A, 3A)$	120
M_{12}	12	$(2A, 2A, 2A, 2A, 2B)$	2048
	12	$(2A, 2A, 2A, 2A, 3A)$	2784
	12	$(2A, 2A, 2A, 2A, 4B)$	7296
M_{11}	12	$(2A, 2A, 2A, 2A, 3A)$	2376
$L_2(11)$	11	$(2A, 2A, 2A, 2A, 2A)$	704
$L_3(2)$	7	$(2A, 2A, 2A, 2A, 2A, 2A)$	1680
	7	$(2A, 2A, 2A, 2A, 3A)$	216
	7	$(2A, 2A, 2A, 2A, 4A)$	192